THE
MATHEMATICAL GAZETTE.

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LOCAL BRANCHES.

LONDON BRANCH.

THE first meeting of the London Branch was held on Jan. 29th at the Polytechnic, Regent Street. There was a large attendance, about 220 being present. The first part of the meeting was devoted to the discussion of the constitution of the Branch and the election of officers. The following were elected officers for 1910: *Chairman*, Mr. C. S. Jackson: *Vice-Chairmen*, Dr. T. P. Nunn and F. J. W. Whipple; *Treasurer*, Mr. W. M. Roberts; *Hon. Secretaries*, Miss H. K. Scott and Mr. P. Abbott; *Committee*, Misses A. E. Bennett, B. M. Cane, B. Hewett, S. Hunt, M. M'Afee, H. M. Sheldon, and C. M. Waters; Dr. Macaulay, J. V. H. Coates, G. F. Daniell, M. Gheury, W. Knowles, W. E. Paterson, Howard Smart, and J. Stanton Wise. The chief business of the meeting was a discussion on the Board of Education circular so far as it referred to the teaching of Geometry. This was opened by Mr. A. W. Siddons, and an interesting discussion followed.

NOTICE.

Members of the Association who desire to be registered as members of the London Branch are asked to send an intimation of their wish to P. Abbott, 5 West View, Highgate Hill, N. (provided they have not already done so). They will then receive all notices, etc., of the Branch.

NORTH WALES BRANCH.

A MEETING of this branch was held on Feb. 19 at Bangor, thirteen members being entertained by Mr. Ferguson, Physics Lecturer at Bangor University College. The president and secretary-treasurer were re-elected, and it was decided not to appoint any committee at present.

Prof. Bryan, the president, read a paper on Examinations, in which he complained that while a large number of Societies and Journals were devoted

to discussions on teaching, no medium exists for considering the best way of conducting examinations, though teaching is largely controlled by them. There are probably several hundred examining bodies in this country, each drawing up elaborate instructions to examiners and corresponding regulations for candidates, but no organised attempt is made to coordinate these bodies. In Welsh schools the difficulty of preparing pupils for different examinations with different ideals is very great. The highest object of education ought to be that of training citizens to perform their duties efficiently and to earn their own livelihood. The efficiency of a national system of education is thus measured by the salaries which it enables people to earn, with a few exceptional cases where men are trained to perform services, not of direct commercial value, such as scientific research. The object of every examination should be to supplement the teacher's efforts in developing the qualities most necessary to make a successful man. Some educationists would abolish examinations and throw the whole responsibility on the teacher, giving him power to grant certificates to satisfactory pupils, but (1) the teacher has enough responsibility already in modelling his teaching so as to deal with the practical needs of pupils, and the whole duty of testing pupils would be too much for one man; (2) the personal element would enter into the question, making it hard to refuse a pupil who works well but is wanting in ability.

The following points can be tested by an examination:

1. The pupil's power of memorising, the importance of which is, however, apt to be much exaggerated. It is difficult to make examinations in some subjects a test of much else than memory, but it should be avoided in mathematics; it is on this account that scholars who are good at examinations so often show little ability in practical work.

2. His clear understanding of work. Bad mistakes should be severely dealt with, but the difficulty is that examination conditions are not conducive to the clear thinking that is necessary.

3. His power of expressing ideas intelligibly, a point in which engineering students are often careless.

4. His power of applying knowledge practically, of knowing the right thing to do in an emergency, of accomplishing a reasonable amount of work accurately in a definite time, and of physical endurance, involving coolness, nerve and presence of mind.

Then as to the nature of examination papers set:

1. Difficult papers mean careless work and lenient marking. From easy papers marks should be deducted for careless work.

2. Long papers mean a race against time and slovenly work. Shorter papers are better, allowing candidates plenty of time for thinking what they are writing, and for correcting mistakes.

3. If choice of questions is allowed, the candidate wastes time in reading all the questions, and very likely finds out when too late that he has attempted wrong ones.

In the ensuing discussion, Mr. Williams (Bethesda County School) said that choice of questions introduced the element of luck too much; Miss Griffiths (Girls' County School), Mr. Botting (Normal College), and others, agreed that papers should contain a certain proportion of easy questions (not beyond the scope of candidates), and some difficult ones in order to differentiate between very good candidates when order of merit was the object. Mr. Ferguson said that the best papers were sent in by those who were allowed a choice of questions, and thought that a syllabus ought not to be too rigidly fixed, as it would limit the best teachers too much. Mr. Turner (Friars' School) considered that a fixed syllabus was necessary, and that for young boys short questions should be set, and pleaded for greater uniformity in "leaving" examinations. Mr. Bartindale thought the harder pieces of book work should be set as well as riders and problems. Other

points referred to were "open book" examinations, in which mathematical tables and even text-books were allowed; the best methods of marking papers, whether extra marks should be given to very good answers, or to a whole paper when well answered; whether there should be a fixed and unalterable maximum of marks for a whole paper, or a fair average mark taken instead. Dr. Bryan spoke of the importance of continuity in papers when external examiners are changed. Votes of thanks to Dr. Bryan and Mr. Ferguson terminated the proceedings.

Mr. Williams will continue this subject at the next meeting at Llanberis on May 28.

THE BOARD OF EDUCATION CIRCULAR ON THE TEACHING OF GEOMETRY.*

THERE are few things more distasteful to the people of this country than a Government Department. It is supposed that from such a place we must expect neither light nor leading. How far this is a just view I need not now discuss: it is equally true that when a Government department *does* give a strong lead, it is condemned for outrunning public opinion. If not King Log, then King Stork.

A case in point is the Board of Education circular on Geometry, a strong pronouncement which has been received by some people with alarm. The Board circular is inspired by clear and constructive ideas. It attempts to crystallize in a new shape the notions that have been set in solution by the discussions of the last few years. Strong solvents were needed before the modern schoolboy could be freed from the system framed for Greek University students. The Board found the process of solution in an advanced state; it has now given a much-needed lead to the process of reconstruction.

Why do we teach geometry? The Board avoids this question, though it lays down that a training in rigid deduction is an essential element in school geometry. *An* essential element: apparently then not the only essential element.

Is it not, in fact, unphilosophical to seek a single *raison d'être* for each subject in the curriculum? If subjects could be justified on such simple grounds education would be an easier thing than we find it. But human affairs are complex. We teach geometry because it has been taught for a long time; and *vis inertiae* decrees that we shall go on teaching it. Probably I shall be putting this view in a more acceptable form if I say the reasons for our present curriculum are historical.

We cannot label each subject, as if it were a drug, with the name of its particular virtue or efficacy. Geometry has long been recommended for the sake of the logical training it imparts. But this claim needs a little examination. Do we really believe that a person who has not studied geometry is likely to be deficient in reasoning powers. It may be so, but I cannot say that I take this view. Must we not limit the expression "reasoning powers" before we confer the monopoly upon geometry? Will a person talk grammatically who has not studied grammar? will a man make a good speech if he has not studied rhetoric? Very possibly—Yes. At the same time, if he has studied grammar and rhetoric, his conversation and his oratory will be on more orthodox lines. Logic was a school study in the middle ages, and produced the type of reasoning used by Shakespeare's clowns. But perhaps at the same time it had a disciplinary effect on the mind. A man who has no geometry may reason correctly. But a course of

* A lecture delivered by Mr. C. Godfrey, Headmaster of the R.N. College, Osborne, before the Oxford Delegacy for Training of Teachers.

geometry would have trained him in the use of a particular logical form—the nearest approach to formal logic that our schools now teach.

What is the peculiar value of this formal training? What chastening, disciplining effect has it on the mind? I should be one of the last to suggest that there is no such good effect. But to me this is a subject of great obscurity. What is the effect of drill on a soldier? Cannot a man walk without learning the goose-step? At first sight, ceremonial drill would seem to be a clumsy way of teaching a man to perform simple actions stiffly. But the experience of all military men forbids us to assume that drill is useless to the soldier. It is formal training.

I can distinguish at least three separate currents in educational ideals: the formal ideal, the Herbartian ideal, and the utilitarian ideal. With regard to the utilitarian ideal, I shall only say here that this is the ideal that appeals most strongly to boys; and, that as long as this is so, we shall do well not to ignore it.

Then there is the formal ideal, the ideal of mental gymnastic: this has affinities with the classical as opposed to the romantic ideal in literature and art. The learner is subjected to drill in some stereotyped form; the Euclidean drill, the grammatical drill, the military drill. The subject-matter of drill may be remote from real life; a logical process seldom used, a dead grammar, and so forth. But this is not held to impair the fortifying value of the exercise. One accepts this ideal loyally and piously, as the educational light of centuries; loyally, but without enthusiasm. And yet it has its enthusiasts. From this ideal flows the cult of formal geometry, a cult which still sways the great majority of the best and most up-to-date teachers of mathematics. We have still to regard training in the use of a logical form as the unique element in geometry teaching, though we must beware of claiming for geometry any monopoly among educational subjects in the development of reasoning power in the wider sense.

A second educational ideal is what I shall, subject to correction, call the Herbartian ideal. By this I understand the ideal that seeks to implant in the mind fruitful ideas—ideas which will find a congenial soil in the boy's existing knowledge and interests. This ideal has been at work powerfully in the recasting of geometry teaching. By its belief in the fertility of ideas, it has emphasized the value of geometrical knowledge as opposed to the study of logical form. Let the boy be thoroughly at home with a new fact or property before he begins to apply formal logic to it. To attain this familiarity, do not reject at any stage the help of experiment, and the recourse to common objects and experience. Geometrical experiment may use models, frameworks, machines; but there is a limit to the amount of apparatus that is convenient. We rely, therefore, in the main upon figures; freehand sketches, where a sketch will reveal the fact that we are looking for; accurate figures, where eye and hand alone are not clever enough. Hence the amalgamation of geometrical drawing with geometrical theory, subjects once divorced, to the great loss of both.

There is a certain educational value in measurement pure and simple, but this is soon exhausted. After the earliest stage we do not make boys draw or measure for the sake of drawing or measuring, but for the sake of the geometrical truths that we are trying to discover. I cannot emphasize this point too strongly. Time has been wasted by setting boys to draw and measure without an adequate aim in view; and I should like to turn aside for a moment to deal with this matter.

Drawing is legitimate as an experimental exercise in leading up to a new theorem, though it must be remembered that a freehand sketch may sometimes be as useful as an accurate figure, and boys should not be allowed to consider themselves helpless without instruments. Again, accurate drawing is generally desirable in solving a problem of construction. The whole point of Euclidean constructions is that they are to be performed with ruler

and compass, and nothing else. Unless the performer has to carry out the work with these actual instruments, he cannot grasp their possibilities and their limitations.

On the other hand, to draw an accurately finished figure in writing out a proposition is sheer waste of time. The figure must be large and neat; straight lines must look straight; parallels parallel and right angles right. But all this can be done freehand and quickly, with the help of the compass perhaps for a circle.

To draw a figure simply for the sake of measuring it and getting marks *may* be waste of time. But let us here do a little clear thinking. Say that the master has given numerical data for a certain construction, numerical in order that all the class may be at work on exactly the same problem. The boys have drawn the figure accurately. How is the master to check it and mark it? He may have an accurately drawn figure on tracing paper, and test with this. But this is probably a sinful waste of master's time. Generally, there can be found some test line in the figure. If the boy measures this, the master may estimate the accuracy of the whole figure well enough by comparing the measurement with the correct answer. Measurements of this kind are not aimless, but a real convenience; and a good many people have quarrelled with such measurement problems as this without giving enough consideration to the matter.

Drawing is not to be abandoned at a definite epoch in the geometry course; practice and theory should advance hand in hand. There is a certain number of teachers who would have a preliminary practical course, all drawing and no thinking; and a subsequent theoretical course, all thinking and no drawing. I admit that this is an unfair way of stating their position; but I cannot agree with this school of thought, and I am glad to find myself confirmed by the Board of Education. The circular says: "The importance of practical work varies from point to point, rising highest where a new idea has to be effectively assimilated. Thus, when the conceptions of locus, envelope, ratio, similarity are first introduced, the practical work should expand. Apart from this, once the earliest stages of the subject are passed, the practical work is of less value, and in the first instance at least attention should be chiefly directed to the development of geometrical and logical power."

The expression "geometrical and logical power" brings me back to the distinction between the formal and the Herbartian ideals from which I started, and for which I quite expect to be taken to task. We have to recognise that "Geometrical power" is not the same as "Logical power." Geometrical power is the power we exercise when we solve a rider. Solving riders was to the average Euclid-trained schoolboy of the past almost as high a flight as turning out a copy of verses was to myself. Why was this so? He had the logical power but had not developed geometrical power.

Riders cannot be solved by logic alone. And unless a boy knows that he can tackle a fairly easy rider with good chance of success, there is no zest in his work. He will not enjoy the sense of mastery which is the true reward of schoolwork done faithfully. To my mind this has always been the keystone of the whole problem—How to make a boy do his work with zest. If you can get him to work with zest, then he can digest your formal training. But formal training, with or without marks and prizes, will not produce zest in the average boy.

There is no subject which can more readily be made exciting to a boy than geometry, if one goes to work in the right way. There must be a good foundation of practical work, and recourse to practical and experimental illustration wherever this can be introduced naturally into the later theoretical course. Only in this way can the average boy develop what I will call the geometrical "eye": the power of seeing geometrical properties detach themselves from a figure. A French geometrician has given us an

account of how he makes his geometrical discoveries. He always works without figures; he sees the figure in his head; he finds that the best way of solving a problem is to go to a concert, or take a seat on the top of a bus, and close his eyes. Well, this is the way of the born geometrician. The average English boy is not a born geometrician, and is perhaps less capable of dealing with abstractions than other boys. He wants all the material help that he can get; given this, most English boys can become very fair, or at any rate, very interested geometricians. Perhaps there is something particularly material and concrete in the English intellect; hand and eye must co-operate with brain in order to produce.

A friend of mine who had not taught mathematics for some years past took a mathematical class for a term recently, and his criticisms were interesting as showing the result of the new methods of teaching. He noticed that the boys could see much more in a figure than he used to find formerly. They attacked deductions with more confidence, and with fair success. On the other hand, he found that they were not so good at writing out propositions.

This last criticism did not surprise me. Boys used to be very ready at writing out propositions; and no wonder, for they did little else. The weak candidate in responsions always chose to take geometry in preference to algebra. However hopeless he was at mathematics, he could always make sure of getting his propositions by heart. The whole thing was overdone: perhaps nowadays it is underdone.

The list of essential propositions is now very moderate; the Cambridge Little-Go schedule requires only about 40 theorems in the first three books; most of these easy. There is no reason why boys should not by the age of sixteen be able to write these out. If the advice of the Board of Education is taken, the list will be further reduced by the omission from the list of several formal theorems of Book I.

The theorems that the Board proposes to arrive at by induction rather than deduction fall into four groups, referring respectively to angles at a point, parallel lines, angles of triangle and polygon, congruence of triangles.

We are recommended to arrive at these facts, not by deduction from two geometrical axioms, but inductively from experience. For instance, how shall we justify the statement that two triangles are congruent if three sides of the one are equal to three sides of the other? Take three white sticks of different lengths; and three green sticks respectively equal to the white sticks. Make a triangle of the three white sticks, and make another triangle of the three green sticks. Who can doubt that these triangles are congruent? But don't be too sure about it. Make a quadrilateral of four white sticks; and another quadrilateral of four equal green sticks. You won't find these two quadrilaterals necessarily congruent. Why then were the triangles congruent? Essentially because three sticks make a rigid framework, and four sticks do not. If you look at Euclid's original proof you will find that all it amounts to is that three sticks make a rigid framework.

"These fundamental propositions are those on which all the subsequent deduction depends, and the essential thing in regard to them is not to analyse them and reduce them to the minimum number of axioms, or, rather, postulates (which is Euclid's method), but to present them in such a way that their truth is as obvious and real to the pupil as the difference between white and black, or between his right hand and his left. Any process which interferes with this directness of vision and apprehension is vicious, whatever claim it may have to logical value, and avenges itself in gross mistakes in subsequent work, due to haziness or lack of grasp of the fundamental facts which have been so laboriously 'proved.'

"With beginners, then, Euclidean proofs of these propositions are out of place, and attention must be concentrated not on formal proofs but on vivid presentation, and accurate, firm apprehension of the propositions themselves."

After these facts have been grasped, the Board contemplate that subsequent work shall be on the well-established lines familiar to all good teachers. "Henceforward, though intuition and experience should be largely used to discover propositions, rigid deductive proof on the basis of the fundamental propositions defined above must be insisted upon."

In effect, the Board argues as follows. We want boys to know geometry, and we want them to build it up deductively. But we can please ourselves as to what are the foundations of this deductive building. Euclid's foundations consisted of certain axioms, two of which were geometrical: his geometrical foundation was therefore very narrow. We advocate a *broad* geometrical foundation, nothing less, in fact, than the four groups of facts mentioned above.

What will be the effect of this circular? Presumably it will have an immediate effect on the schools controlled by the Board. But the effect in the long run will probably depend on the attitude of the teaching profession towards the proposals put forward. And the point that teachers will want to satisfy themselves about is whether the adoption of these proposals will impair the value of geometry as a training in deductive logic.

It may be admitted at once that the number of propositions to which deductive logic is applied will be diminished. But this in itself will not be a serious objection; we want quality rather than quantity. And even as to quantity, we may gain at one end as much as we lose at the other. Oxford and Cambridge require the substance of three books of Euclid for responsions and Little-Go, and this amount has seemed no small matter to many candidates at the age of 18. But the Board says that, in their experience, some schools find it possible to cover effectively in a single year the substance of Euclid Books I. and III. However this may be, few teachers will doubt that any boy of 16 ought to have mastered this amount of geometry with ease, if treated as the Board suggests.

The question then reduces itself to the following. Will the training derived from a strict deductive treatment of the remaining propositions be impaired by the inductive character of the arguments used to establish the fundamental theorems? Now who can suppose that this is a real danger? Was the Euclidean training neutralised by the fact that the boy had never reasoned deductively before he began Euclid? Surely there is no antagonism between inductive and deductive reasoning, as if between an acid and an alkali. The two modes of thought are complementary, and must be *combined*, not isolated, for any fruitful purpose.

One Euclidean ideal has been sacrificed, and we may regret it: the aesthetic ideal of developing the great structure of geometry by pure logic from the minute germ of Euclid's axioms. A great deal of work has been done during the last half century in examining the foundation of geometry. It is work of the greatest philosophical difficulty, and there is as yet no agreement as to what are the fundamental axioms. But there is agreement on one point—that Euclid's axioms are not the true fundamental axioms of geometry. The Euclidean ideal then has been destroyed, not by the Board of Education, but by the labours of pure mathematicians.

A practical advantage of the new system will be the short circuiting of all difficulties as to sequence of propositions. These difficulties are confined almost entirely to the sequence of the fundamental propositions. The controversy is never-ending, for the reason that no sequence that can be proposed is free from some objection, practical or theoretical; and each person has to judge for himself which is the least of the evils before him. There is certainly some inconvenience in the existence of various sequences, though I believe that the inconvenience has been exaggerated. On the other hand, it would be an educational disaster should the outcry about sequence lead to the stereotyping of a new sequence. Well, the trouble melts away if we agree to establish the fundamental propositions

inductively, referring each of them straight back to experience and intuition.

To recapitulate, it is recommended that early geometry teaching shall be in three stages.

The First Stage aims at instilling the primary concepts, and the meaning of geometrical terms. It is not intended to give accuracy in the use of instruments; here instruments are used to help ideas. The meaning of geometrical terms is not taught by definition, but by properly planned series of experiments and questions bearing on everyday objects.

The Second Stage establishes, informally, four groups of fundamental facts, these facts to serve as a foundation for the deductive course that is to follow. By the end of this stage, the pupil must be perfectly familiar with these facts; but he need not have them in any definite order, as each fact is referred straight back to experience. He should be able to quote each fact in good set terms; either in the terms of the book, or in an intelligent variation of his own.

Incidentally, this stage is the time to teach accurate drawing; practice being obtained by drawing triangles, etc., to data; or by problems on heights and distances.

The Third Stage is essentially Euclid revised. We arrive at Euclid's goal, but not by Euclid's road.

ELEMENTARY MATHEMATICS IN EVENING SCHOOLS.

THOUGH the *Mathematical Gazette* contains frequent articles of interest to the pure mathematician, and the Mathematical Association has addressed itself particularly to questions as to the order and method of teaching mathematics in Secondary Schools, yet the *Gazette* contains little of special interest to those concerned in the work of the evening classes, which are so important and special a feature of our British system of education. And yet no inconsiderable proportion of the amount of mathematical teaching in England is given in these evening classes. Its volume may be gauged to some extent by the numbers annually examined by the Board of Education. Thus in 1909 nearly 6000 candidates were examined in Pure Mathematics, 6500 in Practical Mathematics, and 3500 in Practical Plane and Solid Geometry. The number actually attending the classes is of course much greater than the number examined. It is at least two or three times as great, without counting the large number of younger students engaged in the study of mathematics of a lower grade than is catered for by the examinations of the Board of Education. Take the case of a technical school in a certain Lancashire town of 50,000 inhabitants. There are 650 students, about 300 of whom are "technical" in the narrower sense of the term, the rest taking commercial, art, language, or domestic classes. Of these 300 students, 200 are this session taking classes in some stage or other of mathematics, and probably the remaining hundred have taken mathematics classes in a former session. Large numbers of these students do not sit for examination in Mathematics—*e.g.* only 41 actually took the Board of Education's examination last year—because their main subject is an engineering, textile, or building one, and they attend mathematics in order to acquire the necessary knowledge of the subject requisite for working out the problems that arise in these industries. This is merely an illustration to show the large amount of mathematical teaching given in evening schools and the impossibility of gauging it simply by the number of students who present themselves for examination.

The human material for these classes is almost entirely derived from ex-elementary scholars. Of the 619,000 evening students given in the last report of the Board of Education, only 91,000, or 1 in 6·8, were stated to

have attended secondary schools. The proportion is of course far greater in London (1 in 3·8) and in the County Boroughs (1 in 5·6) than in the Counties (1 in 9·1). As then the teachers in the Technical Schools have generally only an elementary school education as far as Standard VI. or VII. on which to build, it has been found necessary to introduce an intermediate course between the elementary and the technical schools, and it is the nature and working of this course that I wish to describe.

This course is known as the Preliminary Technical Course. It has been developed chiefly in the industrial centres of Lancashire, Yorkshire, and Cheshire. Its object is to provide a course of instruction in elementary mathematics, practical geometry, physics, and English composition that will serve as an introduction to all subsequent courses of technical instruction, whether it be in engineering, building, textile, mining, or other industries. It aims at catching the boy when he leaves day school at 13 or 14, giving him a two-years' course in the subjects mentioned, and then passing him on to a technical school at 16 with sufficient mental equipment to grasp at once the specialised instruction provided for him there. An examination is held at the end of each year of the course in order to decide the student's fitness to pass forward. To pass the examination a certain minimum of marks in each subject and a certain minimum total must be obtained. Most of the evening continuation schools and smaller technical schools in Lancashire and Cheshire now hold classes in this course. Last year in these two counties 4000 students were examined in the first year and 2000 in the second, although these examinations were instituted only in 1908. The numbers will be much greater this year.

The syllabus of this examination (conducted by the Union of Lancashire and Cheshire Institutes) will interest those engaged in the teaching of elementary mathematics. It is too long, of course, to be given here *in extenso*, but it may be found in the *Seventieth Report of the Union of Lancashire and Cheshire Institutes*, price 3d., or by post 7d. (Mr. C. Sever, 40 King St. West, Manchester).

The syllabuses are accompanied by a prefatory note which is most important, inasmuch as it contains a brief exposition of the methods of teaching to be employed. This note runs:

"Practical Mathematics and Practical Drawing should, where possible, be taught as one subject and by the same teacher. For the most part each mathematical rule or principle should be considered from the arithmetical, the algebraical, and the geometrical points of view. In the elucidation of a problem, calculation and drawing should always when possible go hand in hand. In the examination, drawings will be asked for in the paper on Practical Mathematics, and calculations will be set in the paper on Practical Drawing.

"The work done in class should as far as possible be based upon the students' own drawings and measurements of actual objects. Students should be provided with a 12-in. ruler (graduated into tenths of an inch, and into centimetres), compasses, set squares, protractors, and calipers. Simple geometrical models and objects, *e.g.* cone, square prism, hexagonal prism, triangular prism, cylinder, sphere, etc., will be found convenient. A supply of small machine parts, *e.g.* nuts, bolts, washers, etc., will also be useful.

"In the examination only those rules and topics which are of real importance to the technical student will be dealt with. For this reason complicated fractions, recurring decimals, G.C.M., and L.C.M. will be excluded. A knowledge of weights and measures which are not in common use will not be required. The questions in algebra will as a general rule be such as are of direct use in solving geometrical, scientific, and technical problems. In Geometry experimental illustrations rather than deductive proofs will for the most part be looked for."

The recommendation that Practical Mathematics and Drawing should be taught by the same teacher is most important, inasmuch as there is necessarily much overlapping in the syllabuses of these subjects. Indeed the whole course should be, and often is, taken by the one teacher. The geometrical models referred to are very largely used. They are wooden models of a convenient size, *e.g.* the cylinder will be about 5 inches high and its diameter about $2\frac{1}{2}$ inches. They are supplied in quantity, so that each student (or at least each pair of students) will have his model before him from which to measure, sketch, calculate areas and volumes, and make plans, elevations, sections, etc. The uses of these models are numerous, *e.g.* calculation of areas, volumes, etc., furnishes ready cases of approximations; the cylinder leads to the evaluation of π, etc. Other objects of less simple construction, *e.g.* nuts and bolts, washers, steel bearings and small machine parts generally are freely used, and such things as woodwork models, inkwells, bottles, etc., are pressed into service. The calipers are handled freely, rough dimensioned sketches quickly made, and calculations and finished drawings follow after. All this takes place while the students are seated at the ordinary class room desks. For the practical science wall cupboards are provided, the flaps of which fall down and form the working benches, while the cupboards contain the balance, bunsen burner, and other apparatus. Failing this the work is carried out on trestle tables.

After two years' experience I can say that the boys take quite readily to the work. They enjoy the handling of instruments and rough apparatus. Their mathematics is far more real to them, their interest is maintained, and they make more rapid progress than of old. There can be no doubt that the evening schools of Lancashire and Yorkshire are now providing a really good training in elementary mathematics and laying a secure foundation for later work. It is by no means rare to find boys in the second year of this course using logarithm tables quite readily, although in this they are transcending their syllabus. The gain of time if every student could handle logarithms will appeal to all who have been engaged in teaching technical classes.

All this may perhaps be considered a chronicle of "small beer," yet it is probably not without interest to many readers of the *Gazette*. There must be many members of the Mathematical Association who are engaged in teaching evening classes and many more who are interested in their work. Even to those engaged solely in secondary schools the way in which the teaching of elementary mathematics to boys who have just left the elementary schools has been tackled in evening classes may appeal, inasmuch as they also have the same product to handle. One noteworthy feature is that the teaching of evening classes has begun to react on the day schools, and many elementary schools are now setting themselves to cover the early part at all events of this preliminary technical course in their top standards, and are following the methods of teaching which are proving so successful in the evening classes. H. T. Holmes.

A SCHOOL COURSE IN MATHEMATICS IN THE XVIIth CENTURY.

The History of the Teaching of Elementary Geometry has recently formed the subject of an interesting work published in America by Dr. A. W. Stamper. In it, in one place, I find a statement that there is no mention of geometry being taught in England outside the Universities during the sixteenth and seventeenth centuries, and in another place it is said that it may be safely guessed that the place of Euclid amongst English schoolbooks dates at the earliest only from the middle of the eighteenth century.

Dr. Gow, the Headmaster of Westminster, also makes the latter remark in his *History of Greek Mathematics*, and adds that he can find no useful information on the mathematical curriculum of an English public school before 1750.

We have, however, some information about the mathematical teaching at one English public school—Christ's Hospital, London—in the seventeenth century, which may be worth giving in detail, though it should be stated at once that the circumstances there were peculiar, and it is not typical of the work at a normal public school at that time.

At that school there were special provisions, on the foundation of Charles II., for teaching mathematics and the elements of navigation to boys who intended to enter the royal navy. The King, after establishing this department, directed Sir Jonas Moore to compile a text-book for the boys. Moore died before he had finished it, but it was completed by his sons-in-law, W. Hanway and J. Potenger, with the aid of Mr. Perkins, then the chief mathematical master in the school, Flamstead, and Halley. The whole was published in 1681, and contains sections on Arithmetic, Algebra, Practical Geometry, together with the substance of Euclid's *Elements*, i-vi, xi, xii, Plane and Spherical Trigonometry, Cosmography, Navigation, the Sphere, Astronomical and Logarithmic Tables, and Geography.

The work had a considerable reputation, and is excellent of its kind, but we may infer from the correspondence described below that, in fact, it was not used in the school. This may have been due to the death of Mr. Perkins in 1680 or 1681. A few years later the question of the school course in mathematics, admittedly insufficient, was raised, and it was determined at the same time, taking advantage of the opportunity, to connect or combine with the King's foundation another one in the school, due to the liberality of a Mr. Stone. The new scheme, with a statement explaining what changes it introduced, was sent to Isaac Newton, and his criticisms thereon invited. A draft of his reply and some of his memoranda on the subject are preserved among the Portsmouth papers.

The syllabus previously in use has not been preserved, but from the extant papers it would seem that at least it required the boys to study the earlier parts of the first book of Euclid's *Elements*; the 10th, 11th, and 12th propositions of the sixth book; and to learn arithmetic. The course lasted two years. Perhaps this represented all the theoretical mathematics then normally prescribed to the King's Scholars, for Newton notes the following omissions. There was no "symbolic arithmetic," *i.e.* no use was made of algebraical symbols and methods. There was no "taking of heights and distances and measuring of planes and solids." There was nothing "of spherical trigonometry, though this was requisite for many problems in astronomy, geography, and navigation." Nothing was taught of "Mercator's chart" or "of computing the way of a ship," or "of longitude, amplitude azimuths, and variations of the compass." And, lastly, there was no "word of reasoning about force and motion, though it be the very life and soul of mechanical skill and manual operations." He continues, "by these defects it's plain that the old scheme wants not only methodizing, but also an enlargement of the learning." Of course the specific criticisms were made with reference to the fact that the boys looked to a career in connection with the sea.

The scheme drawn up to replace this, and submitted to Newton, was designed to occupy two years and a half. He at once wrote cordially approving the plan of working the two foundations in conjunction. He said he thought it would be desirable to keep the number of boys in the King's School constant by promoting to it boys from Stone's Foundation as vacancies occurred, but pointed out that any way of combining the two foundations must be tested by the results, though experience shows "that children thrive best in great schools where there are a competent number of boys in every

form to create an emulation and whet one anothers parts." It is unnecessary to describe this scheme at length, because on the advice of Newton it was considerably modified and extended by the addition of mechanics.

Of the final scheme there are various drafts, differing in minute details. I think we may take the following as being substantially that ultimately approved for the abler boys. The subjects of study were to be Arithmetic; Algebra; Plane and Solid Geometry, practical with rule and compass as well as theoretical; Plane Trigonometry; Drawing and Designing; Instruments and their use; Cosmography, including therein the rudiments of astronomy and the art of making maps and charts; the use of Spherical Triangles; and Mechanics. Also applications of the above to sea problems, laying down courses, and determining positions. Some schedules of the extent to which the subjects should be read are given, but perhaps the above description will suffice.

Newton's letter sent with the final draft, embodying his criticisms, covers eight or more closely written foolscap pages. The general line of his comments is as follows. He begins by a detailed examination of the scheme submitted to him, and discusses the order in which the divisions of the different subjects should be taught. Apart from defects here indicated, he objects to its limitations as encouraging working by rule of thumb, for thus it "comprehends little more than what a child may easily learn by imitation . . . without understanding . . . the reason of what he does," and puts "an industrious blockhead on the same level as a child of parts." Moreover it does not assist a boy "in inventing new things and practices . . . or in judging what comes before him," which should be the aim of education. He considers, however, that, with his amendments and with the addition of mechanics, the new scheme would be satisfactory.

Next he emphasizes the great importance of teaching theoretical mechanics to the boys. No doubt, he says, practical mechanics is also necessary, but the principles of the subject must be understood if one is to be able to deal with new problems. He asserts that the French engineers were superior to those in the British service, and that this was due to the fact that training in France included theoretical mechanics, though since the schools in that country contained a mixture of all sorts of capacities he believes that lads turned out from Christ's Hospital, which draws "children of the best parts selected out of a great multitude," should excel them. He raises the question whether the addition of mechanics will take up too much time, and says that the addition of six months to the two years and a half allowed for the course submitted to him would permit its inclusion, and adds that it may be more for the boys' "advantage to spend this half year at school in an important part of learning which they cannot get at sea, than at sea in learning what they will afterwards learn more readily if well instructed at school before they go thither," though, if the extra time cannot be allowed, he thinks that, perhaps, they may run through all parts of it in two years and a half, but not so well. And I would advise that they should rather be allowed three full years than be sent away smatterers in their learning."

He expresses opinions that in any case it was important that the masters should be able to reckon on a definite period for which the school course would last, and that boys ought not to be allowed to leave their courses incomplete whenever vacancies occurred for which they were eligible; that the school course should be terminated by a public examination, but that the results of private examinations should be regarded by the Governors as confidential; and that it was desirable that boys who had left the school should be encouraged to keep in touch with it by sending to it accounts of their observations, etc.

As far as text-books are concerned, he approves the use of the *Synopsis Algebraica* (no doubt that compiled by J. Alexander, 1693), Ward's *Trigonometry*, and Euclid's *Nova Methodo*, which had been suggested to him, though

as to the last named work he would prefer that the boys should read Euclid in the original, and he suggests that at any rate the abler boys should read, in Barrow's English translation, books i-vi, xi, xii. As for the doctrine of the sphere he recommends the use in the school of the first book of Mercator's *Astronomy*. He sees no disadvantage, indeed rather the reverse, in using books written in Latin.

At Christ's Hospital the boys were receiving a special training, so the mathematical course was taken to a higher standard than at other public schools at that time. This must be borne in mind, and there can be no doubt that the mathematical work at public schools was usually far less than that here described, but I think as a definite detailed example of a school course at the time it is worthy of record, and that Newton's comments on it are of interest.

W. W. ROUSE BALL.

Trinity College, Cambridge, *January*, 1910.

MATHEMATICAL NOTES.

310. [K. 4. b. α.] *Squared paper solution of the equation* $a\cos\phi+b\sin\phi=c$.

The problem may be stated thus: Rectangular axes are turned through the angle ϕ; the new coordinates of a certain point are (b, a), and its old ordinate was c: find ϕ.

Mark in a figure the point $P(b, a)$, and draw the line $y=c$. Let Q, Q', be the points where this line cuts the circle through P with the origin as centre. Then the two values of ϕ are POQ, POQ'. This leads directly to the important relations $\cos\frac{\phi_1+\phi_2}{2}\Big/a=\sin\frac{\phi_1+\phi_2}{2}\Big/b=\cos\frac{\phi_1-\phi_2}{2}\Big/c$, which

(i) enable us to solve the equation by tables, $\frac{\phi_1+\phi_2}{2}$ being the auxiliary angle;

(ii) correspond to the formulae in the quadratic $ax^2+bx+c=0$:

$$x_1+x_2=-b/a\,;\ x_1x_2=c/a\,;$$

(iii) enable us to write the equation

$$\cos\left(\phi-\frac{\phi_1+\phi_2}{2}\right)=\cos\frac{\phi_1-\phi_2}{2},\quad\text{or}\quad\sin\frac{\phi-\phi_1}{2}\sin\frac{\phi-\phi_2}{2}=0.$$

A. C. DIXON.

311. [K. 20. b.] *Fundamental Trigonometrical Expansions.*

Discussions in the *Gazette* have shewn clearly that the use of logarithms of complex quantities is not by any means so simple as it seemed when we learned the proof of Gregory's series in the days of rough and ready analysis. No doubt the simplest proof is the original one by integration, but it is worth while to notice that a very short algebraical proof can be given without appealing to any result more advanced than De Moivre's Theorem.

By equating the imaginary parts in $\cos n\theta+\iota\sin n\theta$ and $(\cos\theta+\iota\sin\theta)^n$, we find that as long as $-\frac{\pi}{4}\leqq\theta\leqq\frac{\pi}{4}$ and $n<1$,

$$\frac{\sin n\theta}{n}=\cos^n\theta\left[\tan\theta-\frac{(n-1)(n-2)}{2.3}\tan^3\theta+\frac{(n-1)(n-2)(n-3)(n-4)}{2.3.4.5}\tan^5\theta-\ldots\right].$$

Now $\mathrm{L}_{n\to0}\frac{\sin n\theta}{n}=\theta$, and the remainder after any number of terms of the series is less than the last term retained. Accordingly it is proved that

$$\theta=\tan\theta-\frac{\tan^3\theta}{3}+\frac{\tan^5\theta}{5}-\ldots.$$

As another example of the method we can investigate the value of the series,

$$C=x\cos\alpha-\frac{x^2}{2}\cos 2\alpha+\frac{x^3}{3}\cos 3\alpha+\ldots.$$

If
$$\frac{\cos\phi}{1+x\cos\alpha}=\frac{\sin\phi}{x\sin\alpha}=\frac{1}{(1+2x\cos\alpha+x^2)^{\frac{1}{2}}}\equiv\frac{1}{R},$$

then
$$R^n\cos n\phi=1+nx\cos\alpha+\frac{n(n-1)}{2}x^2\cos 2\alpha+\ldots.$$

This identity may be written

$$\frac{R^n-1}{n}-\frac{R^n(1-\cos n\phi)}{n}=x\cos\alpha+\frac{n-1}{2}x^2\cos 2\alpha+\frac{(n-1)(n-2)}{2.3}x^3\cos 3\alpha+\ldots.$$

By proceeding to the limit, we find

$$\tfrac{1}{2}\log(1+2x\cos\alpha+x^2)=x\cos\alpha-\frac{x^2\cos 2\alpha}{2}+\frac{x^3\cos 3\alpha}{3}+\ldots.$$

The last series and the conjugate one include all the examples which are usually discussed in connection with complex logarithms. Greater freedom in arranging a course on higher trigonometry is obtained by adopting the alternative method which is sketched in this article. F. J. W. WHIPPLE.

312. [A. 1. a.] I append an elementary and direct method of finding $\sum_1^n n(n+1)$, assuming only the formula for the sum of an A.P.

Consider the terms

1	2	3	4	5	...		n
1	2	3	4	5		$n-1$	
1	2	3	4	5	$n-2$		
⋮							
1	2	3					
1	2						
1							

Adding horizontally the sum is

I. $$\frac{n(n+1)}{2}+\frac{(n-1)n}{2}+\frac{(n-2)(n-1)}{2}\ldots+\frac{1.2}{2}.$$

Adding vertically the sum is

II. $$\begin{aligned}&n+2(n-1)+3(n-2)+4(n-3)\ldots+(n-1)n\\&=1(n-0)+2(n-1)+3(n-2)\ldots+n\{n-\overline{n-1}\}\\&=n(1+2+3\ldots+n)-\{1.2+2.3\ldots+(n-1)n\}.\end{aligned}$$

Now call $1.2+2.3+3.4\ldots+n(n+1)=x$.

Then
$$\frac{x}{2}=\frac{n^2(n+1)}{2}-\{x-n(n+1)\},$$

where the left-hand side is I. and the right-hand II., *i.e.*

$$\begin{aligned}x&=n^3+n^2-2x+2(n^2-n),\\3x&=n(n+1)(n+2),\\x&=\frac{n(n+1)(n+2)}{3}.\end{aligned}$$

This method can be indefinitely extended so as to find

$$\sum_1^n n(n+1)(n+2),\ \sum_1^n n(n+1)(n+2)(n+3)\ldots$$

in succession. S. ANDRADE.

313. [D. 2.] The following proof that $\left(1+\frac{1}{n}\right)^n$ tends to a finite limit as n (an integer) tends to infinity requires only the two simple inequalities:

(a) $\frac{a-x}{b-x}>\frac{a}{b}$, if $a>b>x$. a, b and x being positive.

(b) $(1+\alpha)^n>1+n\alpha$, if n is a positive integer and $-1<\alpha<1$.

We have to show (i) that $\left(1+\frac{1}{n}\right)^n$ increases as n increases, and (ii) that it is always finite.

(i) Consider $\dfrac{\left(1+\frac{1}{n+1}\right)^{n+1}}{\left(1+\frac{1}{n}\right)^n}$, which may be written

$$\left(1+\frac{1}{n+1}\right)^{n+1}\left(1-\frac{1}{n+1}\right)^n; \quad i.e. \quad \left(1+\frac{1}{n+1}\right)\left(1-\frac{1}{(n+1)^2}\right)^n.$$

Now $$\left(1+\frac{1}{n+1}\right)\left(1-\frac{1}{(n+1)^2}\right)^n>\left(1+\frac{1}{n+1}\right)\left(1-\frac{n}{(n+1)^2}\right), \quad \text{.....(b)}$$

$$i.e. \quad >\frac{(n+1)^3+1}{(n+1)^3},$$

$$i.e. \quad >1,$$

which proves (i).

(ii) $$\left(\frac{n+1}{n}\right)^n<\frac{n+1}{n}\cdot\frac{n+\frac{1}{2}}{n-\frac{1}{2}}\cdot\frac{n}{n-1}\cdots\frac{n+1-\frac{n-1}{2}}{n-\frac{n-1}{2}} \quad \text{.....(a)}$$

$$<\frac{2\left(2+\frac{1}{n}\right)}{1+\frac{2}{n}}$$

$$<4.$$

We may prove in a similar manner that $\left(1-\frac{1}{n}\right)^{-n}$ decreases as n increases.

For $$\frac{\left(1-\frac{1}{n+1}\right)^{-n-1}}{\left(1-\frac{1}{n}\right)^{-n}}=\left(1+\frac{1}{n}\right)^{n+1}\left(1-\frac{1}{n}\right)^n$$

$$=\left(1+\frac{1}{n}\right)\left(1-\frac{1}{n^2}\right)^n$$

$$<\frac{1+\frac{1}{n}}{\left(1+\frac{1}{n^2}\right)^n}, \quad \text{since } 1-\frac{1}{n^2}<\frac{1}{1+\frac{1}{n^2}},$$

$$<\frac{1+\frac{1}{n}}{1+\frac{n}{n^2}} \quad \text{.....(b)}$$

$$<1.$$

Also, (ii) above will suggest a proof that $\left(1-\frac{1}{n}\right)^{-n}>\frac{9}{4}$ J. LISTER.

REVIEWS.

Methodologisches und Philosophisches zur Elementar-Mathematik, von G. MANNOURY, Priv.-Doz. f.d. logischen Grundlagen der Mathematik an der Universität zu Amsterdam. Pp. viii, 279. (Haarlem, P. Visser Azn., 1909.)

One of the most important facts that have been discovered in recent years is the fact that the basis of pure mathematics is logic, and logic alone. This does not, of course, mean that what is popularly called 'intuition' plays no part in mathematical discovery; there are certainly interesting psychological problems raised by the consideration of methods of mathematical discovery, but the proposition that the ultimate premises of mathematics are all of a general logical nature has been proved to be the case as rigidly as it can well be by modern mathematicians. This is particularly interesting to mathematicians, since for centuries philosophers have vainly tried to prove or disprove this; and now only mathematicians, like M. Poincaré, who deliberately refuse to study symbolic logic, and who only follow the work of others in it with polemical intentions, still hold a position something like Kant's.

Herr Mannoury's book is an outcome of yearly courses of lectures he has given to teachers of mathematics since 1906; space, time, number and mathematical method are discussed in it; and the controversial literature of the last ten years on Kantianism in mathematics, which has appeared in *Revue de Métaphysique et de Morale*, is especially noticed.

The book has two parts, devoted to the foundations of arithmetic and of geometry respectively. Strangely enough, an account of mathematical logic (pp. 129-154) is reserved for the second part, and a psychological discussion—on which I feel incompetent to form any judgment beyond feeling that it is irrelevant—precedes a logical discussion, beginning on p. 39 with finiteness and infiniteness treated in the orthodox mathematical way. On p. 49, note, Herr Mannoury has the curious remark that Mr. Russell's (1903) contradiction is a merely verbal one.

The author lavishes praise on Peano's logic; but we do not find that he has mentioned what is the most important character of Peano's calculus,—that, unlike Aristotelian logic, it deals with implications between propositional *functions* (containing variables), which are the elements of all mathematical reasoning.

An important work mentioned (p. 75) is that of C. S. Peirce (1881) on the principle of induction. This is often overlooked.

On pp. 109 sqq. Dirichlet is often misprinted for Dedekind.

The book seems to me to be well worth consulting, but to be very incomplete as regards the information that is necessary for fruitful discussion of the questions with which it is concerned. PHILIP E. B. JOURDAIN.

An Introduction to the Study of Integral Equations. M. BÔCHER. (Cambridge Tracts in Mathematics and Mathematical Physics, No. 10.)

Prof. Bôcher's tract is rather different in character from any of its predecessors in the Cambridge Series. Most of these have dealt with subjects selected from "classical" theories, with the general outlines of which every mathematician is familiar. The general theory of Integral Equations, on the other hand, is a product of the last ten years; it is still possible to be a mathematician of the highest reputation and to know next to nothing about it.

It has hitherto been very difficult for anyone who has not kept pace with the development of the theory from the beginning to make good his deficiencies afterwards. The subject is intrinsically difficult, and has been approached from different points of view, which have at first sight but little in common. The literature is scattered, and new contributions appear almost every month. It has been a severe handicap to anyone who wished to "get up" the subject that there has been no connected account of those parts of the theory which may be regarded as tolerably complete. At the same time, so much still remains confessedly incomplete that a treatise on the subject would probably be superseded almost as soon as it was written, and it is not surprising that no one has been willing to write one. The publication of a "Tract" is an ideal compromise, and Prof. Bôcher's should be one of the most valuable of the series.

The most striking characteristic of Prof. Bôcher's exposition is the closeness with which he follows the historical order of the development of the theory. Here, we think, he has probably been wise, though the method is one that has its disadvantages as well as its advantages. The advantages are obvious; the historical order is always the most interesting and the easiest to follow, and the reader is better able to preserve his sense of proportion when different sides of the theory are presented to him in turn. The chief disadvantage is that, when the author's space is as strictly limited as it is here, and when so much of it is devoted to the work of the pioneers, the more systematic modern theories are apt to receive less than their due. And in this tract we certainly wish that less space had been given to Volterra and more to Hilbert and his followers, and in particular that room had been found for some general account of the connection between Integral Equations and Differential Equations, and of the application of this theory to expansions such as Fourier's—to us at any rate the most remarkable and interesting of all its applications.

Still, it would be unreasonable to expect everything in 70 pages, and the wonder is that Prof. Bôcher has been able to give us so much as he has without compressing his argument beyond the limits of intelligibility. He is, indeed, considering the amount of information he contrives to give us in so short a space, extraordinarily lucid and readable throughout, and almost succeeds in making a difficult subject seem easy; and it is in many ways an excellent thing that he should have given us so full an account of the work of Abel and the other precursors of the theory—work which those who are familiar only with the writings of the German school might be in danger of forgetting.

May we make one remark of a technical character with reference to the formula (p. 6, f.n.)

$$\frac{d}{dx}\int_a^x \psi(x-\xi)\phi(\xi)\,d\xi = \int_a^x \psi(x-\xi)\phi'(\xi)\,d\xi,$$

where $\psi(x-\xi)$ is a function such as $(x-\xi)^{-s}$ $(0<s<1)$ and $\phi(a)=0$? The simplest method of procedure seems to be to put $x-\xi=u$ before differentiating, when the result follows at once. This method also leads to the more general formula

$$\frac{d}{dx}\int_a^x \psi(x-\xi)\phi(\xi,\,x)\,d\xi = \psi(x-a)\phi(a,\,x) + \int_a^x \psi(x-\xi)\left\{\frac{\partial\phi}{\partial\xi}+\frac{\partial\phi}{\partial x}\right\}d\xi.$$

G. H. HARDY.

Les systèmes d'équations aux dérivées partielles. By C. RIQUIER. Pp. xxvii+590. Price 20 fr. 1910. (Paris, Gauthier-Villars.)

This book treats the theory of systems of partial differential equations from Cauchy's standpoint, the author's main object being to establish the existence of a set of solutions which can be expressed in the form of power series. The initial conditions by which the solutions are specified are of the same general type as those used in the existence theorems of Cauchy and Mme. Kowalewsky.

By following up an idea introduced by Méray in 1880, M. Riquier has succeeded in defining a class of systems of partial differential equations, called completely integrable, for which the solutions in the form of power series are specified *uniquely* by the initial conditions and for which the series converge when the arbitrary functions and constants occurring in the initial conditions and equations are subject to certain inequalities.

A useful distinction was drawn by Méray between principal variables and derivatives on the one hand, and parametric variables and derivatives on the other. The distinction may be illustrated by taking a single equation,

$$\frac{\partial^2 u}{\partial t^2} = \psi\left(\frac{\partial^2 u}{\partial x^2},\ \frac{\partial u}{\partial x},\ \frac{\partial u}{\partial t},\ u\right)$$

with the initial conditions

$$u=f(x),\quad \frac{\partial u}{\partial t}=g(x) \text{ for } t=0.$$

In this case t is called a principal variable, x a parametric variable. The reason for this is that the initial values of a large number of derivatives depending on x may be calculated at once from the initial conditions; these derivatives are called

parametric. The initial values of the other derivatives must be calculated from the differential equation and from the system of equations which are derived from it by differentiation. These derivatives are the principal ones.

The system of equations which is obtained from the original system by repeated differentiations with regard to the different variables is called the prolongment of the original system. In proving the existence theorem it is necessary to replace these equations by an equivalent system resolved with regard to the principal derivatives. A condition that this should be possible is that certain Jacobians should not vanish. The possession of this property of resolubility is the essential characteristic of a passive system of partial differential equations. A second property is that the resolution should be unique. A completely integrable system of equations is a passive system in which certain inequalities are satisfied and for which the existence theorem can be completely established. The first four chapters of the book are devoted to the general theory of functions of several variables. Chapter VIII. deals with the properties of implicit functions, and contains proofs of some general theorems which are often required, for instance: If the $(\mu+1)$ functions

$$f_1(t_1, \dots t_n), \quad f_2(t_1, \dots t_n), \dots f_\mu(t_1, \dots t_n), \quad f(t_1, \dots t_n),$$
$$\mu < n,$$

can be expanded as power series in the neighbourhood of certain initial values, and one at least of the Jacobians of the first μ functions has a value different from zero, and all the Jacobians of the $\mu+1$ functions are zero, then f can be expressed in terms of $f_1, f_2, \dots f_\mu$.

The author gives some applications of his theory to the transformation of quadratic differential forms, and considers systems of partial differential equations which are reducible to completely integrable forms and other systems which are reducible to total differential equations.

The book contains much that is new, but it is of an advanced nature, and deals with a branch of the subject of partial differential equations which is barely mentioned in other treatises. We can recommend it to an expert but not to a mathematician with only a moderate knowledge of partial differential equations and the modern theory of functions. H. BATEMAN.

Leçons sur la théorie de la croissance. E. BOREL. 1910. (Gauthier-Villars.)

It is now twelve years since the appearance of M. Borel's *Leçons sur la théorie des fonctions*, the first of the admirable series of monographs on the Theory of Functions published under his direction. The object of the series was to provide connected and reasonably compact accounts of modern developments in analysis that have not yet found their way into the standard books. They were to occupy a position half-way between the *Traité d'Analyse* and the original memoir. Beyond question the series has been a most brilliant success, and many of its volumes are quite indispensable to any serious student of analysis: one has only to think of M. Lebesgue's *Leçons sur l'intégration*, of M. Baire's *Fonctions discontinues*, of M. Lindelöf's *Calcul des résidus*, or, above all, of M. Borel's own *Fonctions entières* and *Séries divergentes*. We are therefore naturally reluctant to use any language save that of praise in connection with an addition to the series. But we must confess that we are disappointed with M. Borel's latest volume.

Its cardinal defect is that it falls asunder into two almost entirely disconnected parts. In the first 70 pages we have the pure *Théorie de la croissance*—the theory of orders of greatness and smallness, the Infinitärcalcül of Paul du Bois-Reymond—or rather, we have a little of it; and there is no question that it is a fascinating subject, and that much of what M. Borel tells us is very interesting. Then we have two long chapters of analytical and arithmetical applications, and these chapters, too, contain interesting results, and a few that we had not seen before. But they are choked by a mass of work with which almost everybody who reads the book is bound to be perfectly familiar. Is this really the place for yet another account of the ordinary theory of the Gamma-function (pp. 88-104) or the elements of the theory of continued fractions (pp. 127 *et seq.*)? Surely we are entitled to expect something a little more exciting from a writer of M. Borel's reputation and originality. "Les Chapitres d'applications," says M. Borel, "ont été rédigés de manière que leurs résultats essentiels puissent être compris indépendamment du système de notation exposé dans les premiers

Chapitres," and this is only too true. Whether the first chapters had been written or not, the greater part of the last two (which form more than half the book) might have been taken bodily from almost any *Traité d'Analyse*.

The fact is that the system of notation designed to represent the varying rates of increase of different functions, and developed with such detail by M. Borel in the early part of this book, has not, ingenious as it is, yet found any very serious application. It is still something of a "chinoiserie"; all the important results may be stated just as simply and clearly without it. M. Borel represents the rate of increase of x^n by n, of e^x by ω, of $\log x$ by $\frac{1}{\omega}$, of ax^n by $\frac{1}{\omega}a\omega n$, of $e^{\sqrt{(\log x)}}$ by $\omega.\frac{1}{2}.\frac{1}{\omega}$, of $\sqrt{\{12x^7+3\}}$ by

$$\frac{1}{2}.\frac{1}{\omega}.3.\frac{1}{\omega}.e^5.\omega.4.\omega.7,$$

and so on. He makes a distinction between the *identity* and the *equality* of such orders. Thus the order just written is equal but not identical to F_2, the order of the simple function $x^{\frac{7}{2}}$. He introduces brackets to assist him in denoting the orders of functions when they are only approximately known. Thus a function $f(x)$ such that

$$x^{-n+\epsilon}f(x)\to\infty, \quad x^{-n-\epsilon}f(x)\to 0,$$

for any positive ϵ, however small, is said to have the order (n). He studies the differentiation and integration of such orders, and so on. There is much that is interesting, but it does not seem to lead anywhere, and we wish that M. Borel, instead of dwelling on this part of the subject so long, had told us more of the *Infinitärcalcül* as originally developed by Du Bois-Reymond.

We are sorry that we cannot commend this volume with more enthusiasm. It does not maintain the standard of many previous volumes of the series. That is a very high standard, but it is the standard by which M. Borel must expect to be judged.

G. H. HARDY.

Grundzuge der differential und integral Rechnung. By G. KOWALEWSKI. Pp. 452. Price 10 marks. (Teubner, 1909.)

At the present time teachers of mathematics may be divided into three groups, respectively making the formal perfection of the science, its practical applications, and the individuality of the pupil their chief concern.

Between the three controversy has sometimes flared up. Wellington said that the true object of a battle is the peace of the world, and it is certain that the true object of controversy is agreement. There are signs that the somewhat sharp differences of opinion as to the proper way of setting about the calculus may end, not in the miserable compromise of each going his own way, sneering at the others, but in a hearty agreement as to the real order of growth of the subject.

Dr. Kowalewski points out that the period of free and unscrupulous use of the methods of the calculus lasted for more than a century, and we conclude that considerable familiarity with the calculus as a weapon must in a rational scheme precede any attempt to discuss the difficulties which lie at the foundation of the subject. So far the intuitional school, whose main consideration has been the sequence of ideas in a beginner, are unquestionably right.

M. Poincaré in a fascinating paper has described the genesis of some of his discoveries, and has given ordinary folk the satisfaction of learning that the processes of thought are the same in kind for the genius and for the mortal. Your mind is occupied with a subject. Presently *an idea comes into your mind.*

According to the type of your mind will be the value of the idea, but it comes or grows apart from your will: it occurs to you, in the ordinary phrase which is psychologically correct.

The preliminary condition is thinking about the subject in order that the facts of the case may be clearly conceived and simultaneously grasped.

Thus far the intuitionists are right (and the practical men have supported them, though not always consciously for the right reason) in insisting on acquaintance with facts and methods before generalities and the critical discussion of methods.

But now comes the turn of the other side.

It can never be a light matter for a scientific man to make a statement that is not true—and the line between pedantry and precision is not always easy to draw. Why is it very right and our bounden duty to object in the name of accuracy to such an equation as

"Let 5 pigs = £7. 10s."

and yet for academic quibbling to pause at the word function, or at the legality of differentiating an infinite series?

Suppose, with M. Peano, that t is a decimal between 0 and 1. *E.g.* one value of t will be ·123456654... *ad inf.* t is continuous and ranges from 0 to 1.

Take the alternate figures of t as new decimals x and y. Thus for the above value of t,

$$x = \cdot 13564\ldots,$$
$$y = \cdot 2465\ldots.$$

x and y are determined as soon as t is known. Is x a function of t? Is y a function of t? If x and y are each functions of t, how about $\frac{dy}{dx}$?

Professor Kowalewski sets forth a rigorous outline of the calculus. His treatment of the exponential theorem is of special interest to English teachers.

He begins with a discussion of irrational numbers, obtaining $\lim a^{\frac{1}{n}} = 1$, and deducing a definition of a^x for any real value of x, by taking x as the limit of a sequence $x_1, x_2, \ldots$.

It is then shown that the sequences

$$\left(1+\frac{1}{1}\right)^1\left(1+\frac{1}{2}\right)^2 \ldots \text{ are descending}$$

and

$$\left(1+\frac{1}{1}\right)^2\left(1+\frac{1}{2}\right)^3\left(1+\frac{1}{3}\right)^4 \ldots \text{ are ascending,}$$

whence e as the usual limit, and the differential coefficients of $\log x$ and e^x.

The power series are obtained by showing that $\Sigma a_n x^n$ and $\Sigma n a_n x^{n-1}$ have the same radius of convergence.

An intelligent student with a fair working knowledge of the calculus would find Professor Kowalewski's book most interesting; and would realise clearly the value and necessity of investigations which, if presented to him prematurely, would seem tedious and confusing. C. S. JACKSON.

Eléments de la Théorie des Probabilités. By EMILE BOREL. Pp. 191. 8vo. 1909. (Hermann, Paris.)

This treatise on the mathematical principles of probability is of a type which has been common in France at any time during the last hundred years, but which has not at present an English counterpart. The great prestige of Laplace has accounted, no doubt, for the circumstance that the study of probability is a normal part of the training of a French mathematician, and to lecture on it one of the regular duties of a French professor of pure mathematics. At intervals these lectures, sometimes of an advanced and sometimes of an elementary character, are published, and we are presented with an admirably lucid and beautifully arranged account of the mathematical analysis at the hands of a first-rate pure mathematician, who has, however, no real interest in the subject of probability either from the statistical or from the philosophical standpoint.

Amongst works of this class M. Borel's new volume will take a high place. It possesses, to a remarkable degree, the style and mental quality which seem to exist nowhere but in the work of French scientists. Cultivated and enlightened, unwilling to tire or bore the reader even for a moment, sufficiently aware of the logical and statistical difficulties to avoid and not to blunder into them, M. Borel has touched in an outline of the really beautiful architecture of mathematical probability with an instinct which is best described as a blend between those of the artist and of the host. In these qualities he surpasses the earlier treatises with which his most challenges comparison. But while he is free from the rashness of M. Poincaré's improvisations, he lacks the solidity and completeness of Czuber or Bertrand. His book contains, I think, no original contribution of importance to the subject of probability, except perhaps in the chapters on Geometrical Probability. M. Borel argues here that the mathematician can *define* the geometrical

probability that a point M lies on a certain segment PQ of AB as proportional to the length of the segment, but that this definition is *conventional* until its consequences have been confirmed *à posteriori* by their conformity with the results of empirical observation. He points out that in actual cases there are generally some considerations present which lead us to prefer one of the possible assumptions to the others, and maintains that in this way the contradictory conclusions, which have often been reached in this part of the subject, can be avoided.

There is one point in this connection, the subject of Chapter VIII of Book II, which it may be worth while to raise in somewhat greater detail. He discusses in this chapter M. Poincaré's method of evading the usual contradictions of geometrical probability by introducing an arbitrary function $\phi(x)$, so that, instead of assuming the probability of a point's lying within the interval x and $x + dx$ to be the same as that of its lying within any other infinitesimal interval, the probability is designated by the expression $\phi(x)dx$. The importance of the method arises out of M. Poincaré's claim that in certain cases the final solution is independent of the choice of the arbitrary function. The point to which I should like to call attention is this,—that, in the examples of M. Poincaré and M. Borel, $\phi(x)$ is not *any* arbitrary function. They assume, I think, that $\phi(x)$ is a regular function, continuous and with continuous differential coefficients. In the circumstances of the case, when it is even disputable whether $\phi(x)$ has in all cases any definite numerical measure at all, this is a very considerable assumption. At any rate it involves a logical theory of the value of the probability in these obscure and indefinite cases which is not, in my opinion, valid, and which, it may be added, would not, even if it were valid, help us to avoid more than a few of the *primâ facie* contradictions. It is not possible, however, to discuss such fundamental problems in a review. I wish only to point out that these solutions by means of so-called arbitrary functions are not as independent of prior considerations as the authors of them appear to think.

It must be noticed in conclusion that the absence of any allusion to Czuber's ingenious method of *Mittelwerte* is a serious omission in so full a discussion of geometrical probability. J. M. KEYNES.

Exercise Papers in Elementary Algebra. By E. M. RADFORD. Pp. viii, 112. 1909. (Dent.)

Mr. Radford's collection of 700 problems in 100 papers is intended for students entering for various examinations up to the level of the Matriculation and Intermediate Examinations of London University. The term "elementary" refers to the fundamental processes and results up to and inclusive of the Exponential and Logarithmic expansions. The majority of the questions are original. Approximations and graphical methods are very much in evidence. About half the sets of papers will be useful for the better class of students, and even for those intending to compete for University Scholarships. The collection concludes with some dozen papers on book-work, a useful novelty which is likely to save teachers some trouble. The answers are published separately at 6d.

Problem Papers in Mathematics. By R. C. FAWDRY. Pp. 240. 4s. 6d. 1909. (Macmillan.)

Mr. Fawdry's collection fills a gap. Series A consists of 25 papers on Arithmetic, Algebra, Geometry, Graphs and easy Trigonometry. Series B contains the same number, generally more difficult, and including Mechanics. Series C comprises Coordinate Geometry and the Calculus, and one-fifth of its papers are stiff enough to test the best Woolwich candidates. Easy revision and specimen papers are appended. The book will be heartily welcomed, and will meet the requirements of all those who are interested in the examinations held by the Civil Service Commissioners, of teachers in Technical Colleges, and of candidates for Entrance Scholarships at the Universities.

Coordinate Geometry. By J. H. GRACE and F. ROSENBERG. 4s. 6d. 1907. (Clive & Co.)

We hope it is not too late to call attention to Messrs. Grace and Rosenberg's *Coordinate Geometry*, which has been unaccountably overlooked till now, and the authors must accept our apologies for this belated reference to what we must

describe as "what some will consider an ideal text-book for the private student." If it has a fault *qua* text-book, it is that it gives the student hardly any opportunity for going astray. There is too much in the way of guidance and cautions and hints, and we are afraid that in the opinion of some teachers this goes far to impair the value of a text-book, though many may think it is the best book possible for the student who has no one to help him, and who cares for nothing but getting through the London B.A. In fact, we have here in an exaggerated form what many are beginning to be convinced is a distinctly evil tendency in the school-books of the day. The student is thrown too little on his own resources. He can hardly help passing the London B.A., and, if that is all the book aims at, it has achieved its end. The authors have shown themselves past masters in the art of making a difficult subject "easy." "Analytical Geometry contains, confessedly, so many difficulties for the ordinary beginner that any work that develops the subject in a more gradual and more explanatory manner than existing text-books has a good *raison d'être*." In justice to the authors, who, after all, are thoroughly competent mathematicians who have been set to write a work for the budding London B.A., they are perfectly conscious of the position. In connection with the worked-out examples, they say: "We would impress upon the reader the very great importance we attach to an attempt on the reader's part to solve book-work theorems and the illustrative examples for himself previous to reading the proofs given. In most cases the reader will not succeed, but the benefit accruing from the attempt cannot, as all experienced teachers know, be over-estimated." No advice could be more sound, but how many students ever read a preface? and, even if they read it, there is no way of securing that the advice will be taken. The temptation to the ordinary student to make use of what is thrust before his gaze will, we venture to think, be found irresistible. Dogs can be trained to hold the biscuit on the nose, but it is only when under the teacher's eye that they will "wait." It is, however, no use complaining, for, as long as examinations are what they are, such books, we suppose, must be written. And if we are to appraise this volume from the point of view of its immediate aim, the getting the student through a certain examination with the minimum of intellectual effort, *i.e.* so that as little as possible shall he be compelled to *stop and think*, then we can fairly say that it is a notable achievement in the art of exposition. We have detected one misprint. Part II., p. 13 (B), for $(\mathit{fl}-\mathit{gm})^2$ read $(\mathit{fl}-\mathit{gm})^2$.

Examples and Homework in Preliminary Practical Mathematics. Parts I. and II. By T. I. COWLISHAW. In two parts, 1s. 6d. each. 1909. (Longmans, Green.)

The compiler states in his preface that "the primary object of this book is to secure uniformity of instruction and homework in those Preliminary Classes in Practical Mathematics which are so large as to necessitate division into several sections, under a corresponding number of teachers." We confess we are entirely unable to see in what way this aim is attained by a collection of examples, some headed "to be worked in class" and others "homework," in any more marked degree than by the use of other compilations of the kind that come under our notice. Part I. provides a first year's course, and is intended to place the student in a position to begin the ordinary text-book on Practical Mathematics. Part II. is suitable for candidates for Stage II. of the Board of Education examination in Practical Mathematics. In both parts the alternate pages are left blank, so that the student may fill in with notes corresponding to the examples on the opposite pages. "It has long been my opinion that this is the most powerful way in which a text-book of real value to the student can be formed." Here we have the germ of a fruitful idea, which is, we believe, destined to influence class books of the future in more subjects than Mathematics. Part I. contains short sets of additional examples in "textile" and in "chemical" Arithmetic. The examples throughout are varied and well chosen, and may well find a place on the teacher's shelves, even if not adopted for use in his classes. Answers to Part I. may be obtained by teachers from the publishers for 1s. 8d.—rather a prohibitive price, but the compensation comes in the Answers to Part II., which the preface almost seems to imply may be obtained free of charge.

Coordinate Geometry. By H. B. FINE and H. D. THOMPSON. 6s. 6d. net. (Macmillan & Co.)

This is a two-year course, which presents some points well worth the consideration of the teacher. Of the 300 pages, about 190 are devoted to the coordinate geometry of a plane, and the rest to coordinate geometry of space. In looking through the American text-book a British teacher is almost always puzzled as to what previous equipment the reader is supposed to possess. About p. 170 we find sections dealing with the path of a projectile, and the graphs of the trigonometrical functions. At the end of the book we find the radian defined, the trigonometrical ratios explained, and a list of formulae such as $\sin^2 A + \cos^2 A = 1$, $\cos x - \cos y = -2 \sin \frac{1}{2}(x+y) \sin \frac{1}{2}(x-y)$, and the like. We are even given the formulae for the sum and product of the roots of a quadratic equation, and the Remainder theorem is explained. The next point to be noted is the order selected by the authors. In "deference to usage" the chapter on the circle is placed after those dealing with the straight line, but they strongly recommend that the first application of the method acquired by the student in the first two chapters shall at once be applied to new material, *e.g.* to the parabola rather than to the circle, with most of the properties of which the student may be supposed to be familiar. We are inclined to think that there are sound psychological grounds for this plan. At the same time we cannot help feeling that the same grounds combined with other pedagogical considerations would prevent us in practice from relegating loci to the last chapter. The authors themselves are hardly consistent in this respect, for we find questions on loci in the chapters on planes and straight lines, the equations to the conics have been obtained by treating the conics as loci, and a considerable number of the questions at the end of the chapters must almost necessarily be loci-problems in disguise. We would prefer to see more importance attached to the parametric equations, to which one section is given. They might make their appearance each in its own chapter—indeed the tangent at the point $a\cos\phi$, $b\sin\phi$ is actually given in the chapter on the ellipse. Opinion will also differ as to the wisdom of postponing pole and polar properties until after the consideration of the general equation of the conic. In these matters each is guided by personal experience, and as the *Gazette* is intended to be the medium for the exchange of views on just such points as these, we hope that these remarks may elicit from teachers what they have found to be the most effective order to adopt.* We do not in the least intend to convey thereby any depreciation of Messrs. Fine and Thompson's excellent little introduction, which seems to us to be admirably adapted to its purpose. We need only add that the sections dealing with solid geometry are confined to the more simple formulae, and the student should be able, after reading them, to deal in a satisfactory manner with the geometry of the conicoid. A word of praise must be given to the diagrams and the eight or nine photographs of conicoids.

Plane Trigonometry. By H. S. CARSLAW. Pp. xviii+293+xi. 4s. 6d. 1909. (Macmillan.)

According to the sub-title this is intended to be "an elementary text-book for the higher classes of Secondary Schools and for Colleges." It is at once differentiated from its predecessors by the use of four-figure tables from the outset, and by the insistence of the author on the use of squared paper for the purpose of arriving at approximate results, and for checking results found in other ways. About 130 pages bring us to the end of solution of triangles. Circular Measure is the subject of the final chapter of this part of the book. Everything is very clearly and concisely arranged. The only criticism we would venture to make is that most of the examples at the end of Chapter XI. seem at first sight to have become somewhat unwarrantably difficult as compared with those that have gone before. We may draw attention to the discussion in the section, Chapter XII., "Length of a Curve," as an excellent introduction to the idea of the limiting value of an infinite sequence of numbers. With respect to the note at the bottom

* We should not be surprised to find a general desire for complete freedom as to order. What will suit a form in one year will only lead to muddlement in its successors of the next.

of page 145, that "we have no right to speak of the fraction for the *value* $\theta=0$ at all," it would be simpler and more direct to use the notation $\mathrm{Lt}_{\theta\to 0}$.

In Part II. we first have a chapter on the geometrical properties of Triangles and their Associated Circles, completed by sixty examples of which the last thirty will test the powers of the most ambitious of scholarship candidates. We turn with most interest, perhaps, to the chapter on Series. Wisely, we think, the author has excluded all infinite series with imaginary terms. There is an excellent chapter on the Trigonometrical series, and we are glad to see that among the geometrical illustrations those due to Mr. Whipple, and contained in Vol. IV. of the *Gazette*, are included. The series for sine and cosine with their expressions as infinite products bring to a close the second part of this admirably fresh treatment of a well worn subject.

Although a considerable amount of the material to be found in the ordinary text-books has been omitted from these pages, the gain is obvious. We cannot think that the student who has been through this book, thus lightened of the load that at the early stage is unnecessary, and who has worked out a fair proportion of the carefully chosen examples, will be any the less prepared to tackle his "Hobson." At any rate we have not the slightest doubt that there will be many teachers prepared with a light heart to make the experiment.

A School Arithmetic. By H. S. Hall and F. H. Stevens. Pp. xiii+475 +xxxix. 4s. 6d. with answers. 1908. (Macmillan.)

This useful text-book may be had in two parts each at 2s. 6d. and 2s. with and without answers, or the answers to the whole may be had for 1s. The authors are such old hands at the game of text-book construction that there is little need to do more than record that their latest effort is worthy of the attention of every teacher. The collection of examples is more than adequate for any conceivable purpose; they are, as far as we can see, carefully graded; plenty of oral exercises are given in the early stages. Algebraical methods of an easy character are introduced naturally where needed. "Practice" disappears as a separate mysterious process unconnected with anything else, being treated as a natural illustration of multiplication in the chapters on fractions and decimals, and the same remark holds good of ratio, proportion, and percentages. Graphical methods are well to the fore. The use of four-figure tables is explained, and if the student wants to find a cube root he gets it either by factors or by logarithms. Special praise may be awarded to the sections dealing with approximations, which are uncommonly well done. Incidentally we may note that the authors make use of some of the suggestions made by Mr. A. Lodge in the *Gazette* as long ago as 1894. The requirements of the Civil Service examinations are met by the inclusion of the necessary matter dealing with continued fractions. We trust we have said enough to indicate that this book has to be reckoned with by teachers who are attracted by the claims of several other competitors of great merit on the market.

Annuaire pour l'An 1910, avec des Notices Scientifiques. 1 fr. 50 c. (Gauthier-Villars.)

We have noted the usual contents of this useful and ridiculously cheap little volume so often, that we may now content ourselves with referring to the "notices scientifiques" peculiar to the present issue. M. Baillaud gives a full account of the proceedings of the 1909 meeting of the International Permanent Committee appointed to deal with the Astrographic Chart. M. Lallemand discourses (87 pp.) on the tides of the earth's crust and the elasticity of the globe. And M. Bigourdan supplies a valuable chronological list of memoirs in the *Annuaires* since they began to appear in 1804. The memoirs are also tabulated according to names of authors and of subj.

BOOKS, ETC., RECEIVED.

Neue Tafel der Viertelquadrate aller natürlichen Zahlen von 1 *bis* 20000 *zur bildung aller möglichen produkte im bereiche* 1×1 *bis* 10000×10000. By J. Bojko. Pp. 40. 1·50 m. 1909. (Speidel, Zurich.)

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